HISTORICAL INTRODUCTION

In the middle of the eighteenth century there was a prolonged controversy as to the possibility of the expansion of an arbitrary function of a real variable in a series of sines and cosines of multiples of the variable. The question arose in connection with the problem of the Vibrations of Strings. The theory of these vibrations reduces to the solution of the Differential Equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2},$$

and the earliest attempts at its solution were made by D'Alembert,* Euler,† and Bernoulli.‡ Both D'Alembert and Euler obtained the solution in the functional form

$$y = \phi(x+at) + \psi(x-at).$$

The principal difference between them lay in the fact that D'Alembert supposed the initial form of the string to be given by a single analytical expression, while Euler regarded it as lying along any arbitrary continuous curve, different parts of which might be given by different analytical expressions. Bernoulli, on the other hand, gave the solution, when the string starts from rest, in the form of a trigonometrical series

$$y = A_1 \sin x \cos at + A_2 \sin 2x \cos 2at + \ldots,$$

and he asserted that this solution, being perfectly general, must contain those given by Euler and D'Alembert. The importance of his discovery was immediately recognised, and Euler pointed out that if this statement of the solution were correct, an arbitrary function of a single variable must be developable in an infinite series of sines of multiples of the variable. This he

* *Mém. de l'Académie de Berlin*, **3**, p. 214, 1747.

† *loc. cit.*, **4**, p. 69, 1748. ‡ *loc. cit.*, **9**, p. 173, 1753.

held to be obviously impossible, since a series of sines is both periodic and odd, and he argued that if the arbitrary function had not both of these properties it could not be expanded in such a series.

While the debate was at this stage a memoir appeared in 1759* by Lagrange, then a young and unknown mathematician, in which the problem was examined from a totally different point of view. While he accepted Euler's solution as the most general, he objected to the mode of demonstration, and he proposed to obtain a satisfactory solution by first considering the case of a finite number of particles stretched on a weightless string. From the solution of this problem he deduced that of a continuous string by making the number of particles infinite.† In this way he showed that when the initial displacement of the string of unit length is given by $f(x)$, and the initial velocity by $F(x)$, the displacement at time t is given by

$$y = 2\int_0^1 \sum_1^\infty (\sin n\pi x' \sin n\pi x \cos n\pi a t) f(x')\,dx'$$

$$+\frac{2}{a\pi}\int_0^1 \sum_1^\infty \frac{1}{n}(\sin n\pi x' \sin n\pi x \sin n\pi a t) F(x')\,dx'.$$

This result, and the discussion of the problem which Lagrange gave in this and other memoirs, have prompted some mathematicians to deny the importance of Fourier's discoveries, and to attribute to Lagrange the priority in the proof of the development of an arbitrary function in trigonometrical series. It is true that in the formula quoted above it is only necessary to change the order of summation and integration, and to put $t=0$, in order that we may obtain the development of the function $f(x)$ in a series of sines, and that the coefficients shall take the definite integral forms with which we are now familiar. Still Lagrange did not take this step, and, as Burkhardt remarks,‡ the fact that he did not do so is a very instructive example of the ease with which an author omits to draw an almost obvious conclusion from his results, when his investigation has been undertaken with another end in view. Lagrange's purpose was to demon-

* Cf. Lagrange, *Œuvres*, T. I., p. 37. † *loc. cit.*, § 37.

‡ Burkhardt, "Entwicklungen nach oscillirenden Functionen," *Jahresber. D. Math. Ver., Leipzig*, **10**, Hft. II., p. 32, 1901.

strate the truth of Euler's solution, and to defend its general conclusions against D'Alembert's attacks. When he had obtained his solution he therefore proceeded to transform it into the functional form given by Euler. Having succeeded in this, he held his demonstration to be complete.

The further development of the theory of these series was due to the astronomical problem of the expansion of the reciprocal of the distance between two planets in a series of cosines of multiples of the angle between the radii. As early as 1749 and 1754 D'Alembert and Euler had published discussions of this question in which the idea of the definite integral expressions for the coefficients in Fourier's Series may be traced, and Clairaut, in 1757,* gave his results in a form which practically contained these coefficients. Again, Euler,† in a paper written in 1777 and published in 1793, actually employed the method of multiplying both sides of the equation

$$f(x) = a_0 + 2a_1 \cos x + 2a_2 \cos 2x + \ldots 2a_n \cos nx + \ldots$$

by $\cos nx$ and integrating the series term by term between the limits 0 and π. In this way he found that

$$a_n = \frac{1}{\pi}\int_0^\pi f(x) \cos nx \, dx.$$

It is curious that these papers seem to have had no effect upon the discussion of the problem of the Vibrations of Strings in which, as we have seen, D'Alembert, Euler, Bernoulli, and Lagrange were about the same time engaged. The explanation is probably to be found in the fact that these results were not accepted with confidence, and that they were only used in determining the coefficients of expansions whose existence could be demonstrated by other means. It was left to Fourier to place our knowledge of the theory of trigonometrical series on a firmer foundation, and, owing to the material advance made by him in this subject the most important of these expansions are now generally associated with his name and called Fourier's Series.

His treatment was suggested by the problems he met in the Mathematical Theory of the Conduction of Heat. It is to be

* *Paris, Hist. Acad. sci.*, 1754 [59], Art. iv. (July 1757).

† *Petrop. N. Acta.*, **11**, 1793 [98], p. 94 (May 1777).

found in various memoirs, the most important having been presented to the Paris Academy in 1811, although it was not printed till 1824-6. These memoirs are practically contained in his book, *Théorie mathématique de la chaleur* (1822). In this treatise several discussions of the problem of the expansion of a function in trigonometrical series are to be found. Some of them fail in rigour. One is the same as that given by Euler. However, it is a mistake to suppose that Fourier did not establish in a rigorous and conclusive manner that a quite arbitrary function (meaning by this any function capable of being represented by an arc of a continuous curve or by successive portions of different continuous curves), could be represented by the series we now associate with his name, and it is equally wrong to attribute the first rigorous demonstration of this theorem to Dirichlet, whose earliest memoir was published in 1829.* A closer examination of Fourier's work will show that the importance of his investigations merits the fullest recognition, and Darboux, in the latest complete edition of Fourier's mathematical works,† points out that the method he employed in the final discussion of the general case is perfectly sound and practically identical with that used later by Dirichlet in his classical memoir. In this discussion Fourier followed the line of argument which is now customary in dealing with infinite series. He proved that when the values

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx',$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos nx'\,dx', \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\sin nx'\,dx', \end{aligned}\right\} \quad n \geqq 1,$$

are inserted in the terms of the series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

the sum of the terms up to $\cos nx$ and $\sin nx$ is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\frac{\sin \frac{1}{2}(2n+1)(x'-x)}{\sin \frac{1}{2}(x'-x)}\,dx'.$$

He then discussed the limiting value of this sum as n becomes

* Dirichlet, *J. Math., Berlin*, **4**, p. 157, 1829.

† *Œuvres de Fourier*, T. I., p. 512, 1888.

infinite, and thus obtained the sum of the series now called Fourier's Series.

Fourier made no claim to the discovery of the values of the coefficients

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx',$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\cos nx'\,dx', \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x')\sin nx'\,dx', \end{aligned}\right\} \quad n \geqq 1.$$

We have already seen that they were employed both by Clairaut and Euler before this time. Still there is an important difference between Fourier's interpretation of these integrals and that which was current among the mathematicians of the eighteenth century. The earlier writers by whom they were employed (with the possible exception of Clairaut) applied them to the determination of the coefficients of series whose existence had been demonstrated by other means. Fourier was the first to apply them to the representation of an entirely arbitrary function, in the sense in which we have explained this term. In this he made a distinct advance upon his predecessors. Indeed Riemann* asserts that when Fourier, in his first paper to the Paris Academy in 1807, stated that a completely arbitrary function could be expressed in such a series, his statement so surprised Lagrange that he denied the possibility in the most definite terms. It should also be noted that he was the first to allow that the arbitrary function might be given by different analytical expressions in different parts of the interval; also that he asserted that the sine series could be used for other functions than odd ones, and the cosine series for other functions than even ones. Further, he was the first to see that when a function is defined for a given range of the variable, its value outside that range is in no way determined, and it follows that no one before him can have properly understood the representation of an arbitrary function by a trigonometrical series.

The treatment which his work received from the Paris Academy is evidence of the doubt with which his contemporaries viewed

* Cf. Riemann, "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe," *Göttingen, Abh. Ges. Wiss.*, **13**, §2, 1867.

his arguments and results. His first paper upon the Theory of Heat was presented in 1807. The Academy, wishing to encourage the author to extend and improve his theory, made the question of the propagation of heat the subject of the *grand prix de mathématiques* for 1812. Fourier submitted his *Mémoire sur la propagation de la Chaleur* at the end of 1811 as a candidate for the prize. The memoir was referred to Laplace, Lagrange, Legendre, and the other adjudicators; but, while awarding him the prize, they qualified their praise with criticisms of the rigour of his analysis and methods,* and the paper was not published at the time in the *Mémoires de l'Académie des Sciences.* Fourier always resented the treatment he had received. When publishing his treatise in 1822, he incorporated in it, practically without change, the first part of this memoir; and two years later, having become Secretary of the Academy on the death of Delambre, he caused his original paper, in the form in which it had been communicated in 1811, to be published in these *Mémoires.*† Probably this step was taken to secure to himself the priority in his important discoveries, in consequence of the attention the subject was receiving at the hands of other mathematicians. It is also possible that he wished to show the injustice of the criticisms which had been passed upon his work. After the publication of his treatise, when the results of his different memoirs had become known, it was recognised that real advance had been made by him in the treatment of the subject and the substantial accuracy of his reasoning was admitted.‡

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* Their report is quoted by Darboux in his Introduction (p. vii) to *Œuvres de Fourier*, T. I.:—"Cette pièce renferme les véritables équations différentielles de la transmission de la chaleur, soit à l'intérieur des corps, soit à leur surface; et la nouveauté du sujet, jointe à son importance, a déterminé la Classe à couronner cet Ouvrage, en observant cependant que la manière dont l'Auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du côté de la rigueur."

† *Mémoires de l'Acad. des Sc.*, **4**, p. 185, and **5**, p. 153.

‡ It is interesting to note the following references to his work in the writings of modern mathematicians:

Kelvin, *Coll. Works,* Vol. III., p. 192 (Article on "Heat," *Enc. Brit.*, 1878).

"Returning to the question of the Conduction of Heat, we have first of all to say that the theory of it was discovered by Fourier, and given to the world through the French Academy in his *Théorie analytique de la Chaleur*, with

The next writer upon the Theory of Heat was Poisson. He employed an altogether different method in his discussion of the question of the representation of an arbitrary function by a trigonometrical series in his papers from 1820 onwards, which are practically contained in his books, *Traité de Mécanique* (T. I. (2[e] éd.) 1833), and *Théorie mathématique de la Chaleur* (1835). He began with the equation

$$\frac{1-h^2}{1-2h\cos(x'-x)+h^2}=1+2\sum_1^\infty h^n\cos n(x'-x),$$

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solutions of problems naturally arising from it, of which it is difficult to say whether their uniquely original quality, or their transcendently intense mathematical interest, or their perennially important instructiveness for physical science, is most to be praised."

Darboux, Introduction, *Œuvres de Fourier*, T. I., p. v, 1888.

"Par l'importance de ses découvertes, par l'influence décisive qu'il a exercée sur le développement de la Physique mathématique, Fourier méritait l'hommage qui est rendu aujourd'hui à ses travaux et à sa mémoire. Son nom figurera dignement à côté des noms, illustres entre tous, dont la liste, destinée à s'accroître avec les années, constitue dès à présent un véritable titre d'honneur pour notre pays. La *Théorie analytique de la Chaleur* . . . , que l'on peut placer sans injustice à côté des écrits scientifiques les plus parfaits de tous les temps, se recommande par une exposition intéressante et originale des principes fondamentaux ; il éclaire de la lumière la plus vive et la plus pénétrante toutes les idées essentielles que nous devons à Fourier et sur lesquelles doit reposer désormais la Philosophie naturelle ; mais il contient, nous devons le reconnaître, beaucoup de négligences, des erreurs de calcul et de détail que Fourier a su éviter dans d'autres écrits."

Poincaré, *Théorie analytique de la propagation de la Chaleur*, p. 1, § 1, 1891.

"La théorie de la chaleur de Fourier est un des premiers exemples de l'application de l'analyse à la physique ; en partant d'hypothèses simples qui ne sont autre chose que des faits expérimentaux généralisés, Fourier en a déduit une série de conséquences dont l'ensemble constitue une théorie complète et cohérente. Les résultats qu'il a obtenus sont certes intéressants par eux-mêmes, mais ce qui l'est plus encore est la méthode qu'il a employée pour y parvenir et qui servira toujours de modèle à tous ceux qui voudront cultiver une branche quelconque de la physique mathématique. J'ajouterai que le livre de Fourier a une importance capitale dans l'histoire des mathématiques et que l'analyse pure lui doit peut-être plus encore que l'analyse appliquée."

Boussinesq, *Théorie analytique de la Chaleur*, T. I., p. 4, 1901.

"Les admirables applications qu'il fit de cette méthode (*i.e.* his method of integrating the equations of Conduction of Heat) sont, à la fois, assez simples et assez générales, pour avoir servi de modèle aux géomètres de la première moitié de ce siècle ; et elles leur ont été d'autant plus utiles, qu'elles ont pu, avec de légères modifications tout au plus, être transportées dans d'autres branches de la Physique mathématique, notamment dans l'Hydrodynamique et dans la Théorie de l'élasticité."

h being numerically less than unity, and he obtained, by integration,

$$\int_{-\pi}^{\pi} f(x')\frac{1-h^2}{1-2h\cos(x'-x)+h^2}\,dx'$$
$$=\int_{-\pi}^{\pi} f(x')dx' + 2\sum_1^\infty h^n \int_{-\pi}^{\pi} f(x')\cos n(x'-x)\,dx'.$$

While it is true that by proceeding to the limit we may deduce that

$$f(x) \text{ or } \tfrac{1}{2}[f(x-0)+f(x+0)]$$

is equal to

$$\operatorname*{Lt}_{h\to 1}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')\,dx' + \frac{1}{\pi}\sum_1^\infty h^n \int_{-\pi}^{\pi} f(x')\cos n(x'-x)\,dx'\right],$$

we are not entitled to assert that this holds *for the value* $h=1$, unless we have already proved that the series converges for this value. This is the real difficulty of Fourier's Series, and this limitation on Poisson's discussion has been lost sight of in some presentations of Fourier's Series. There are, however, other directions in which Poisson's method has led to most notable results. The importance of his work cannot be exaggerated.*

After Poisson, Cauchy attacked the subject in different memoirs published from 1826 onwards,† using his method of residues, but his treatment did not attract so much attention as that given about the same time by Dirichlet, to which we now turn.

Dirichlet's investigation is contained in two memoirs which appeared in 1829 ‡ and 1837.§ The method which he employed we have already referred to in speaking of Fourier's work. He based his proof upon a careful discussion of the limiting values of the integrals

$$\int_0^a f(x)\frac{\sin\mu x}{\sin x}\,dx \dots a>0,$$
$$\int_a^b f(x)\frac{\sin\mu x}{\sin x}\,dx \dots b>a>0,$$

* For a full treatment of Poisson's method, reference may be made to Bôcher's paper, "Introduction to the Theory of Fourier's Series," *Ann. Math., Princeton, N. J.* (Ser. 2), 7, 1906.

† See Bibliography, p. 303.

‡ *J. Math., Berlin*, 4, 1829.

§ *Dove's Repertorium der Physik*, Bd. I., p. 152, 1837.

as μ increases indefinitely. By this means he showed that the sum of the series

$$a_0+(a_1 \cos x+b_1 \sin x)+(a_2 \cos 2x+b_2 \sin 2x)+\ldots,$$

where the coefficients a_0, etc., are those given by Fourier, is

$$\tfrac{1}{2}[f(x-0)+f(x+0)] \ldots -\pi<x<\pi,$$
$$\tfrac{1}{2}[f(-\pi+0)+f(\pi-0)] \ldots x=\pm\pi,$$

provided that, while $-\pi<x<\pi$, $f(x)$ has only a finite number of ordinary discontinuities and turning points, and that it does not become infinite in this range. In a later paper,* in which he discussed the expansion in Spherical Harmonics, he showed that the restriction that $f(x)$ must remain finite is not necessary, provided that $\int_{-\pi}^{\pi} f(x)\,dx$ converges absolutely.

The work of Dirichlet led in a few years to one of the most important advances not only in the treatment of trigonometrical series, but also in the Theory of Functions of a Real Variable; indeed it may be said to have laid the foundations of that theory. This advance is to be found in the memoir by Riemann already referred to, which formed his *Habilitationsschrift* at Göttingen in 1854, but was not published till 1867, after his death. Riemann's aim was to determine the *necessary* conditions which $f(x)$ must satisfy, if it can be replaced by its Fourier's Series. Dirichlet had shown only that certain conditions were *sufficient*. The question which Riemann set himself to answer, he did not completely solve: indeed it still remains unsolved. But in the consideration of it he perceived that it was necessary to widen the concept of the definite integral as then understood.

Cauchy, in 1823, had defined the integral of a *continuous* function as the limit of a sum, much in the way in which it is still treated in our elementary text-books. The interval of integration (a, b) is first divided into parts by the points

$$a=x_0,\ x_1,\ x_2,\ \ldots\ x_{n-1},\ x_n=b.$$

The sum

$$S=(x_1-x_0)f(x_1)+(x_2-x_1)f(x_2)+\ldots+(x_n-x_{n-1})f(x_n)$$

is formed. And the integral $\int_a^b f(x)\,dx$ is defined as the limit of

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* *J. Math., Berlin*, 17, 1837.

this sum when the number of parts is increased indefinitely and their length diminished indefinitely. On this understanding every continuous function has an integral.

For discontinuous functions, he proceeded as follows:

If a function $f(x)$ is continuous in an interval (a, b) except at the point c, in the neighbourhood of which $f(x)$ may be bounded or not, the integral of $f(x)$ in (a, b) is defined as the sum of the two limits

$$\operatorname*{Lt}_{h\to 0}\int_a^{c-h} f(x)\,dx, \quad \operatorname*{Lt}_{h\to 0}\int_{c+h}^{b} f(x)\,dx,$$

when these limits exist.

Riemann dismisses altogether the requirement of continuity, and in forming the sum S multiplies each interval $(x_r - x_{r-1})$ by the value of $f(x)$, not necessarily at the beginning (or end) of the interval, but at a point ξ_r arbitrarily chosen between these, or by a number intermediate between the lower and upper bounds of $f(x)$ in (x_{r-1}, x_r). The integral $\int_a^b f(x)\,dx$ is defined as the limit of this sum, if such exists, when the number of the partial intervals is increased indefinitely and their length tends to zero.

Riemann's treatment, given in the text in a slightly modified form, is now generally adopted in a scientific treatment of the Calculus. It is true that a more general theory of integration has been developed in recent years, chiefly due to the writings of Lebesgue,* de la Vallée Poussin and Young; that theory is mainly for the specialist in certain branches of Pure Mathematics. But no mathematician can neglect the concept of the definite integral which Riemann introduced.

One of the immediate advances it brought was to bring within the integrable functions a class of discontinuous functions whose discontinuities were infinitely numerous in any finite interval. An example, now classical, given by Riemann, was the function defined by the convergent series:

$$1 + \frac{[x]}{1^2} + \frac{[2x]}{2^2} + \dots \frac{[nx]}{n^2}$$

where $[nx]$ denotes the positive or negative difference between nx and the nearest integer, unless nx falls midway between two consecutive integers, when the value of $[nx]$ is to be taken as

* See footnote, p. 77.

zero. The sum of this series is discontinuous for every rational value of x of the form $p/2n$, where p is an odd integer prime to n.

With Riemann's definition the restrictions which Dirichlet had placed upon the function $f(x)$ were considerably relaxed. To this Riemann contributed much, and the numerous writers who have carried out similar investigations since his time have still further widened the bounds, while the original idea that every continuous function admitted of such an expansion has been shown to be false. Still it remains true that for all practical purposes, and for all ordinary functions, Dirichlet's investigation established the convergence of the expansions. Simplifications have been introduced in his proof by the introduction of the Second Theorem of Mean Value, and the use of a modified form of Dirichlet's Integral, but the method which he employed is still the basis of most rigorous discussions of Fourier's Series.

The nature of the convergence of the series began to be examined after the discovery by Stokes (1847) and Seidel (1848) of the property of Uniform Convergence. It had been known since Dirichlet's time that the series were, in general, only conditionally convergent, their convergence depending upon the presence of both positive and negative terms. It was not till 1870 that Heine showed that, if the function is finite and satisfies Dirichlet's Conditions in the interval $(-\pi, \pi)$, the Fourier's Series converges uniformly in any interval lying within an interval in which $f(x)$ is continuous. This condition has, like the other conditions of that time, since been somewhat modified.

In the last thirty or forty years quite a large literature has arisen dealing with Fourier's Series. The object of many of the investigations has been to determine sufficient conditions to be satisfied by the function $f(x)$, in order that its Fourier's Series may converge, either throughout the interval $(-\pi, \pi)$, or at particular points of the interval. It appears that the convergence or non-convergence of the series for a particular value of x really depends only upon the nature of the function in an arbitrarily small neighbourhood of that point, and is independent of the general character of the function throughout the interval, this general character being limited only by the necessity for the existence of the coefficients of the series. These memoirs—

associated chiefly with the names of Du Bois-Reymond, Lipschitz, Dini, Heine, Cantor, Jordan, Lebesgue, de la Vallée Poussin, Hobson and Young—have resulted in the discovery of sufficient conditions of wide scope, which suffice for the convergence of the series, either at particular points, or, generally, throughout the interval. The necessary and sufficient conditions for the convergence of the series at a point of the interval, or throughout any portion of it, have not been obtained. In view of the general character of the problem, this is not surprising. Indeed it is not improbable that no such necessary and sufficient conditions may be obtainable.

In many of the works referred to above, written after the discovery by Lebesgue (1902) of his general theory of integration, series whose terms did not exist under the old definition of the integral are included in the discussion.

The fact that divergent series may be utilised in various ways in analysis has also widened the field of investigation, and indeed one of the most fruitful advances recently made arises from the discussion of Fourier's Series which diverge. The word "sum," when applied to a divergent series, has, of course, to be defined afresh; but all methods of treatment agree in this, that when the same process is applied to a convergent series the "sum," according to the new definition, is to agree with the "sum" obtained in the ordinary way. One of the most important methods of "summation" is due to Cesàro, and in its simplest form is as follows:

Denote by s_n the sum of the first n terms of the series

$$u_1+u_2+u_3+\dots .$$

Let
$$S_n=\frac{s_1+s_2+\dots+s_n}{n}.$$

When $\underset{n\to\infty}{\text{Lt}}\, S_n=S$, we say that the series is "summable," and that its "sum" is S.

It is not difficult to show that if the series

$$u_1+u_2+u_3+\dots$$

is convergent, then
$$\underset{n\to\infty}{\text{Lt}}\, S_n=\underset{n\to\infty}{\text{Lt}}\, s_n,$$

so that the "condition of consistency" is satisfied. [Cf. § 102.]

Fejér was the first to consider this sequence of Arithmetic Means,

$$s_1, \quad \frac{s_1+s_2}{2}, \quad \frac{s_1+s_2+s_3}{3}, \ldots$$

for the Fourier's Series. He established the remarkable theorem* that the sequence is convergent, and its limit is

$$\tfrac{1}{2}(f(x+0)+f(x-0))$$

at every point in $(-\pi, \pi)$ where $f(x+0)$ and $f(x-0)$ exist, the only conditions attached to $f(x)$ being that, if bounded, it shall be integrable in $(-\pi, \pi)$, and that, if it is unbounded, $\int_{-\pi}^{\pi} f(x)\,dx$ shall be absolutely convergent.

Later, Hardy showed that if a series

$$u_1+u_2+u_3+\ldots$$

is summable by this method, and the general term u_n is of the order $1/n$, the series is convergent in the ordinary sense, and thus the sum $S\ (=\underset{n\to\infty}{\mathrm{Lt}}\ S_n)$ and the sum $s\ (=\underset{n\to\infty}{\mathrm{Lt}}\ s_n)$ will be the same. [Cf. § 102.]

Hardy's theorem, combined with Fejér's, leads at once to a new proof of the convergence of Fourier's Series, and it can also be applied to the question of its uniform convergence. Many of the results obtained by earlier investigators follow directly from the application to Fourier's Series of the general theory of summable series.†

Recent investigations show that the coefficients in Fourier's Series, now frequently called Fourier's Constants, have important properties, independent of whether the series converges or not. For example, it is now known that if $f(x)$ and $\phi(x)$ are two functions, bounded and integrable in $(-\pi, \pi)$, and a_0, a_n, b_n are Fourier's Constants for $f(x)$, and a_0', a_n', b_n' those for $\phi(x)$, the series

$$2a_0a_0'+\sum_1^{\infty}(a_na_n'+b_nb_n')$$

converges, and its sum is $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\phi(x)\,dx$. To this theorem,

* *Math. Ann., Leipzig*, **58**, 1904.

† Chapman, *Q. J. Math., London*, **43**, 1912; Hardy, *London, Proc. Math. Soc.* (Ser. 2), **12**, 1913.

and to the results which follow from it, much attention has recently been given, and it must be regarded as one of the most important in the whole of the theory of Fourier's Series.*

The question of the approximation to an arbitrary function by a finite trigonometrical series was examined by Weierstrass in 1885.† He proved that if $f(x)$ is a continuous and periodic function, given the arbitrary small positive quantity ϵ, a finite Fourier's Series can be found in a variety of ways, such that the absolute value of the difference of its sum and $f(x)$ will be less than ϵ for any value of x in the interval. This theorem was also discussed by Picard, and it has been generalised in recent memoirs by Stekloff and Kneser.

In the same connection, it should be noted that the application of the method of least squares to the determination of the coefficients of a finite trigonometrical series leads to the Fourier coefficients. This result was given by Töpler in 1876.‡ As many applications of Fourier's Series really only deal with a finite number of terms, these results are of special interest.

From the discussion of the Fourier's Series it was a natural step to turn to the theory of the Trigonometrical Series

$$a_0 + (a_1 \cos x + b_1 \sin x) + a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

where the coefficients are no longer the Fourier coefficients. The most important question to be answered was whether such an expansion was unique; in other words, whether a function could be represented by more than one such trigonometrical series. This reduces to the question of whether zero can be represented by a trigonometrical series in which the coefficients do not all vanish. The discussion of this and similar problems was carried on chiefly by Heine and Cantor,§ from 1870 onwards, in a series of papers which gave rise to the modern Theory of Sets of Points, another instance of the remarkable influence Fourier's Series have had upon the development of mathematics.‖ In this place it will

* Cf. Young, *London*, *Proc. R. Soc.* (A), **85**, 1911.

† *J. Math.*, *Berlin*, **71**, 1870.

‡ Töpler, *Wien*, *Anz. Ak. Wiss.*, **13**, 1876.

§ Bibliography, p. 305.

‖ Van Vleck, "The Influence of Fourier's Series upon the Development of Mathematics," *American Association for the Advancement of Science* (Atlanta), 1913.

See also a paper with a similar title by Jourdain, *Scientia*, Bologna (Ser. 2), **22**, 1917.

be sufficient to state that Cantor showed that all the coefficients of the trigonometrical series must vanish, if it is to be zero for all values of x in the interval $(-\pi, \pi)$, with the exception of those which correspond to a set of points constituting, in the language of the Theory of Sets of Points, a set of the n^{th} order, for which points we know nothing about the value of the series.

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